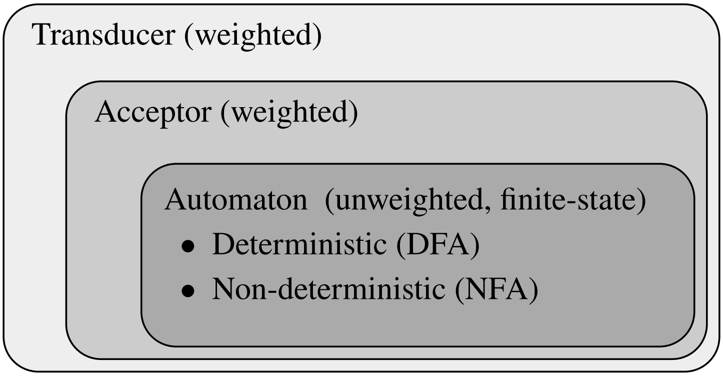


Figure 2.1: A hierarchy of automata classes from general to specific in terms of representation power. Weighted transducers can represent anything that weighted acceptors can represent. Weighted acceptors in turn can represent any unweighted finite-state automata.

2 Acceptors and Transducers

2.1 Automata

The broad class of graphs we are going to look at are finite-state automata. These include deterministic finite-state automata (DFAs) and non-deterministic finite-state automata (NFAs). More specifically we will consider a generalization of DFAs and NFAs called weighted finite-state acceptors (WFSAs). That's a mouthful, so I will just call them *acceptors*. We will also consider a further generalization of an acceptor called a *transducer* (weighted finite-state transducers or WFSTs). Figure 2.1 shows the relation between these three graphs; transducers, acceptors, and automata. Transducers are the most expressive in terms of their representational power, followed by acceptors followed by unweighted automata.

Before we dive into acceptors and transducers, let's introduce some general graph terminology that I will use throughout. In the following graph a *state* or *node* is represented by a circle. The arrows represent connections between two states. We usually refer to these as *arcs* but sometimes also *edges*. The graph is directed since the connections between states are unidirectional arrows. The arcs in a graph can have labels. In figure 2.2 the arc between states 0 and 1 has a label of a. Similarly, the arc between states 1 and 2 has a label of b. The graph is an example of a finite-state automata (FSA) or finite-state machine (FSM), so called because it has a finite number of nodes.

An automata is deterministic if for each state and label pair there is only one outgoing transition which matches that label. An automata is nondeterministic if

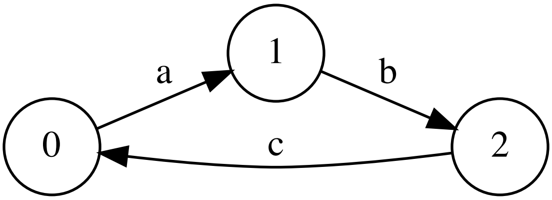
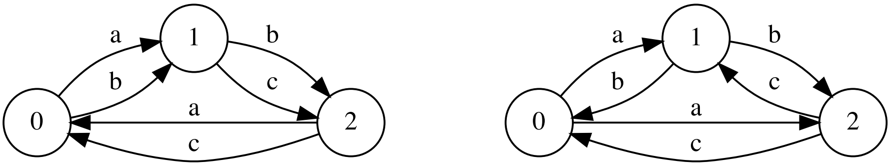


Figure 2.2: An example of a simple finite-state automata.



(a) Deterministic **(b)** Nondeterministic

Figure 2.3: An example of a deterministic automata and a nondeterministic automata. The nondeterministic automata has two arcs leaving state 0 both with label a and two arcs leaving state 2 both with the label c.

multiple transitions leaving a state have the same label. The graphs in figure 2.3 show an example of a deterministic and a nondeterministic automata. In general, acceptors and transducers to can be nondeterministic.

2.2 Acceptors

Let's start by constructing some very basic automata to get a feel for their various properties.

The start state $s = 0$ has a bold circle around it. The accepting state 1 is represented with concentric circles. Each arc has a label and a corresponding weight. So the first arc from state 0 to state 1 with the text $a/0$ means the label is a and the weight is 0. The fact that there is only a single label on each arc means this graph is an *acceptor*. Since it has weights, we say its a weighted acceptor. Since the number of states is finite, some would call it a weighted finite-state acceptor or WFSA. Again, that's a mouthful, so I'll just call these graphs acceptors.

An accepting path in the graph is a sequence of arcs which begin at a start state and end in an accepting state. By concatenating the labels on an accepting path, we get a string which is accepted by the graph. So the string aa is accepted by the

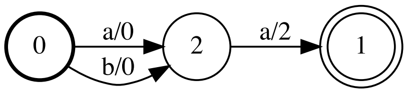


Figure 2.4: An example of a simple acceptor. The label on each arc shows the input label and weight, so the $a/0$ represents a label of a and a weight of 0.

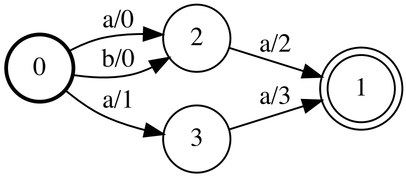


Figure 2.5: An acceptor which has multiple paths for the same sequence, aa.

graph by following the state sequence $0 \to 2 \to 1$. The string ba is also accepted by the graph by following the same state sequence but taking the arc with label b when traversing from state 0 to state 1. The language of the acceptor is the set of all strings which are accepted by it. You may also encounter "recognized" used as a synonym for "accepted". Let the variable $\mathcal{A}$ represent the acceptor in figure 2.4. In general, I'll use uppercase script letters to represent graphs. Let $\mathcal{L}(\mathcal{A})$ denote the language of $\mathcal{A}$. In this case $\mathcal{L}(\mathcal{A}) = \{aa, ba\}$.

There are different ways to compute the weight of a string accepted by the graph. The most common is to sum the weights of the arcs on the accepting path for that string. For example the string aa in the graph in figure 2.4 has a weight of $0 + 2 = 2$. Another option would be to multiply the weights. These two options correspond to interpreting the weights as either log probabilities or probabilities. We'll have more to say about this later.

The graph in figure 2.5 accepts the same sequence by multiple paths.

The string aa is accepted along the state sequence $0 \to 2 \to 1$ and along the state sequence $0 \to 3 \to 1$. In this case, to compute the score of aa we need to consider both paths. Again we have a couple of options here. The most common approach is to *log-sum-exp* the individual path scores. Again this corresponds to interpreting the path scores as log probabilities. We'll use $\text{LSE}(s_1, s_2)$ to denote the *log-sum-exp* of the two scores s_1 and s_2:

$$\text{LSE}(s_1, s_2) = \log\left(e^{s_1} + e^{s_2}\right). \tag{3}$$

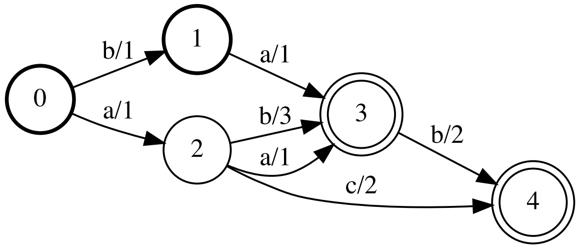


Figure 2.6: An acceptor with multiple start states (0 and 1) and multiple accept states (3 and 4).

So the overall weight for the string *aa* in the graph in figure 2.5 is given by:

$$\log\left(e^{0+2} + e^{1+3}\right) = 4.13.$$

Acceptors can have multiple start states and multiple accept states. In the graph in figure 2.6, the states 0 and 1 are both start states, and the states 3 and 4 are both accept states.

It turns out that allowing multiple start or accept states does not increase the expressive power of the graph. With ϵ transitions (which we will discuss soon), one can convert any graph with multiple start states and multiple accept states into an equivalent graph with a single start state and a single accept state.

Note also that start states can have incoming arcs (as in state 1) and accept states can have outgoing arcs, as in state 3.

Example 2.1. Compute the score of the string *ab* in figure 2.6.

The two state sequences which accept the string *ab* are the states $0 \to 2 \to 3$ and $1 \to 3 \to 4$. The overall score is given by:

$$\log(e^{1+3} + e^{1+2}) = 4.31.$$

■

Graphs can also have self-loops and cycles. For example, the graph in figure 2.7 has a self-loop on the state 0 and a cycle following the state sequence $0 \to 1 \to 2 \to 0$.

The language of a graph with cycles and self-loops contains infinitely many strings. For example, the language of the graph in figure 2.7 includes any string that starts

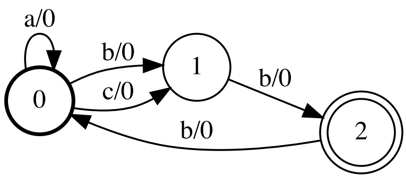


Figure 2.7: A graph with a self-loop on the state 0 and a cycle from $0 \to 1 \to 2 \to 0$.

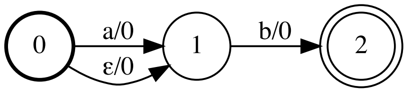


Figure 2.8: An acceptor with an ϵ transition on the second arc between state 0 and 1.

with zero or more as and ends in bb. As a regular expression we write this as a^*bb where the * denotes zero or more as.

The ϵ symbols has a special meaning when it is the label on an arc. Any arc with an ϵ label can be traversed without consuming an input token in the string. So the graph in figure 2.8 accepts the string ab, but it also accepts the string b because we can traverse from state 0 to state 1 without consuming an input.

As it turns out, any graph with ϵ-transitions can be converted to an equivalent graph without ϵ transitions. However, this usually comes at a large cost in the size of the graph. Complex languages can be represented by much more compact graphs with the use of ϵ-transitions.

Example 2.2. Convert the graph in figure 2.6 which has multiple start and accept states to an equivalent graph with only a single start and accept state using ϵ transitions.

The graph in figure 2.9 is the equivalent graph with a single start state and a single accept state.

The construction works by creating a new start state and connecting it to the old start states with ϵ transitions with a weight of 0. The old start nodes are regular internal nodes in this new graph. Similarly the old accept states are now regular states and they connect to the new accept state with ϵ transitions with a weight of 0. ■

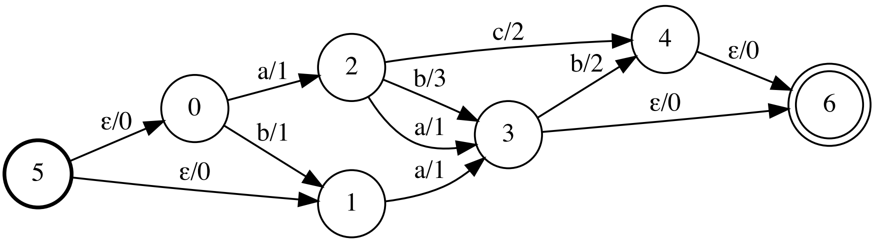


Figure 2.9: The equivalent graph using only a single start state and accept state to the graph in figure 2.6 which has multiple start and accept states.

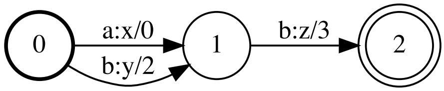


Figure 2.10: An example of a simple transducer. The label on each arc shows the input label, the output label, and the weight. So $a:x/0$ represents an input label of a, and output label of x, and a weight of 0.

2.3 Transducers

A *transducer* maps input strings to output strings. Transducers are a generalization of acceptors. Every acceptor is a transducer, but not every transducer is an acceptor. Let's look at a few example transducers to understand how they work.

The arc labels distinguish an acceptor from a transducer. A transducer has both an input and output arc label. The arc labels are of the form $a:x/0$ where a is the input label x is the output label and 0 is the weight. An acceptor can be represented as a transducer where the input and output labels on every arc are identical.

Instead of saying that a transducer accepts a given string, we say that it *transduces* one string to another. The graph in figure 2.10 transduces the string ab to the string xz and the string bb to the string yz. The weight of a transduced pair is computed in the same way as in an acceptor. The scores of the individual arcs on the path are summed. The path scores are combined with *log-sum-exp*. So the weight of the transduced pair (ab, xz) in the graph in figure 2.10 is $0 + 3 = 3$.

We have to generalize concept of the language from an acceptor to a transducer. I'll call this generalization the transduced set. Since it will always be clear from context if the graph is an acceptor or transducer, I'll use the same symbol $\mathcal{L}$ to represent the transduced set. If $\mathcal{T}$ is a transducer, then $\mathcal{L}(\mathcal{T})$ is the set of pairs

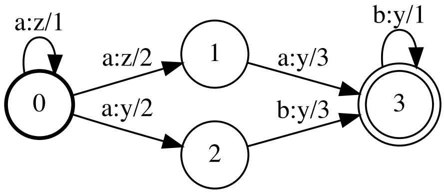


Figure 2.11: An example transducer in which the sequence aab is transduced to the sequence zyy on multiple paths.

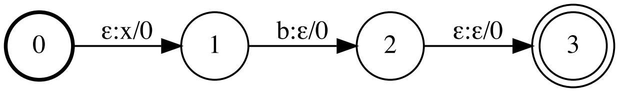


Figure 2.12: A transducer with ϵ transitions. The ϵ can be just the input label, just the output label, or both the input and output label.

of strings transduced by $\mathcal{T}$. More formally, a pair of strings $(\mathbf{x}, \mathbf{y}) \in \mathcal{L}(\mathcal{T})$ if $\mathcal{T}$ transduces $\mathbf{x}$ to $\mathbf{y}$.

Example 2.3. Compute the score of the transduced pair (aab, zyy) in the graph in figure 2.11.

The two paths which transduce aab to zyy are following the state sequence $0 \rightarrow 1 \rightarrow 3 \rightarrow 3$ and $0 \rightarrow 0 \rightarrow 2 \rightarrow 3$. The score of the first path is 6 and the score of the second path is 6. So the overall score is:

$$\log\left(e^6 + e^6\right) = 6.69.$$

■

Transducers can also have ϵ transitions. The ϵ can be either the input label on an arc, the output label on an arc, or both. When the ϵ is the input label on an arc, it means we can traverse that arc without consuming an input token, but we still output the arc's corresponding output label. When the ϵ is the output label, the opposite is true. The input is consumed but no output is produced. And when the ϵ is both the input and the output label, the arc can be traversed without consuming an input or producing an output.

In the graph in figure 2.12, the string b gets transduced to the string x. On the first arc between states 0 and 1, we output an x without consuming any token. On the second arc between states 1 and 2, a b is consumed without outputting any new token. Finally, on the arc between states 2 and 3 we neither consume nor output a token.