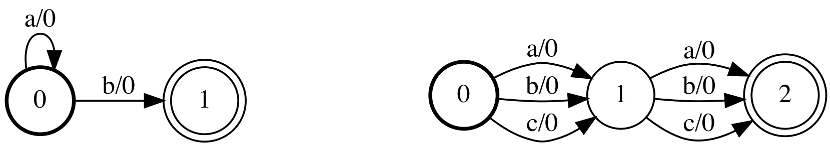


Figure 4.1: The acceptor on the left has a language of a^*b and the acceptor on the right accepts all nine sequences of length 2. The only sequence accepted by both, and hence is in the intersection, is the string *ab*.

4 Advanced Operations

4.1 Intersect

The language of the intersection of two acceptors is the set of strings which is accepted by both of them. The score of a path in the intersected graph is the sum of the score of the path in each of the input graphs. More formally the language of the intersected graph is given by $\{\mathbf{x} \mid \mathbf{x} \in \mathcal{L}(\mathcal{A}_1) \ \wedge \ \mathbf{x} \in \mathcal{L}(\mathcal{A}_2)\}$.

The analogous operation for transducers is the composition, which I will discuss in more detail in the next section. In most machine learning applications, intersect and compose tend to be the primary operations used to compute more complex graphs from simpler input graphs. Thus gaining a deeper understanding of these two operations is worth the time.

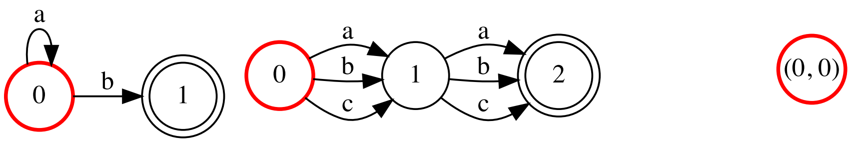
We'll use the two graphs in figure 4.1 to illustrate a general algorithm for computing the intersection. In this case, the intersected graph is easy to see by inspecting the two graphs. The only string which is recognized by both graphs is the string *ab*, hence the intersected graph is the graph which recognizes *ab*.

The general algorithm relies on a queue to explore pairs of states from the two input graphs. Pseudocode is given in algorithm 4.1.

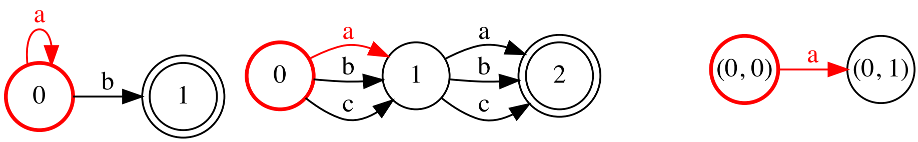
Some of the steps of the intersect algorithm on the two input graphs in figure 4.1 are shown in the sequence of graphs in figure 4.2. The states explored at each step are highlighted in red. The intersected graph is the third graph on the right which is constructed over the steps of the algorithm.

The pseudocode in algorithm 4.1 does not handle ϵ transitions. The challenge with including ϵ transitions in intersect and compose is discussed in section 4.2.1.

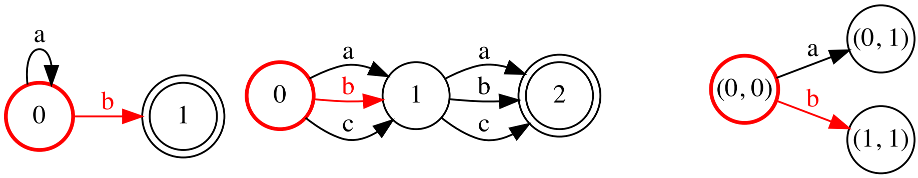
Another potential issue with the algorithm 4.1is that the output graph it constructs is not *trim*. An automata is trim if every state in the graph is part of a path which starts in a start state and terminates in an accept state. In short, the graph



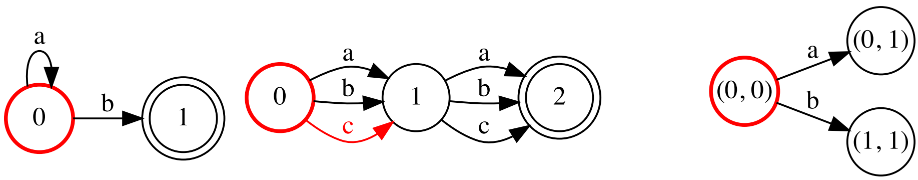
(a) The starts states in the input graphs are 0 and 0. So we add the start state $(0, 0)$ to the intersected graph and to the queue to be explored.



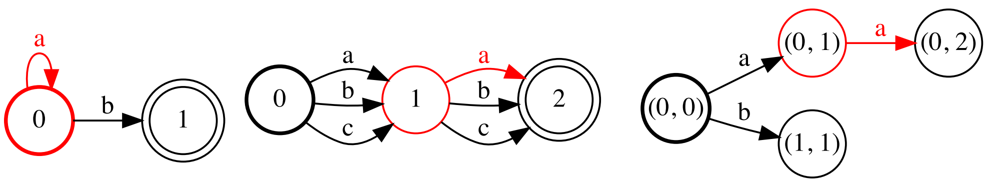
(b) The first pair of outgoing arcs match on the label a. This means the downstream state $(0, 1)$ is reachable in the intersected graph. So we add $(0, 1)$ to the intersected graph and to the queue to be explored.



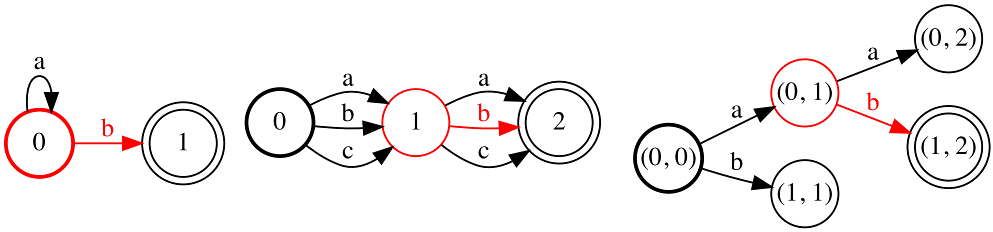
(c) The next matching pair of outgoing arcs match on the label b. Also, we haven't visited the downstream state $(1, 1)$ yet. So we add $(1, 1)$ to the intersected graph and to the queue to be explored.



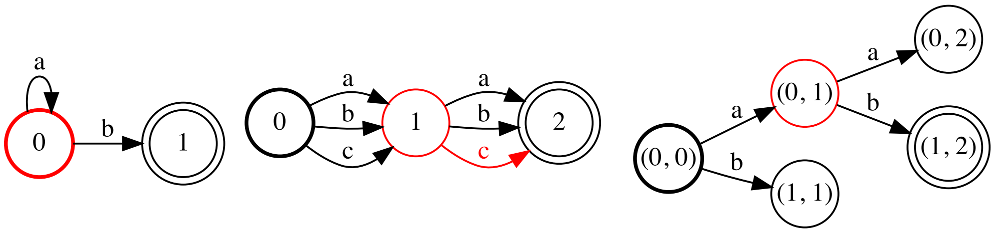
(d) The arc in the second input graph with label c does not have a matching arc from state 0 in the first input graph, so it is ignored.



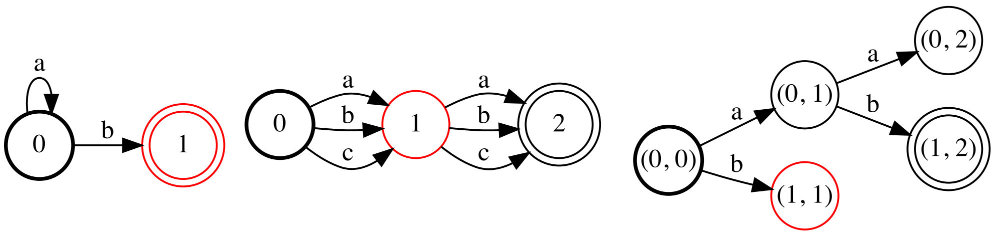
(e) At this point we have considered all the matching outgoing arcs from $(0, 0)$ so we now move on to the next state pair in the queue, $(0, 1)$. From $(0, 1)$, a pair of outgoing arcs match on the label a. The downstream state $(0, 2)$ is new so we add it to the intersected graph and to the queue.



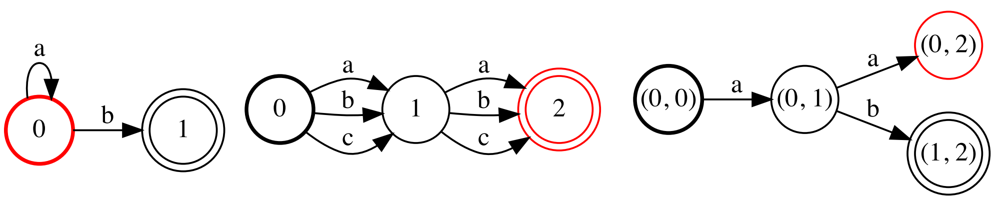
(f) The next pair of matching outgoing arcs match on the label b and lead to the state pair $(1, 2)$. We add $(1, 2)$ to the queue and to the intersected graph. Also, since 1 is an accept state in the first graph and 2 is an accept state in the second graph, $(1, 2)$ is an accept state in the intersected graph.



(g) Again, the arc in the second input graph with label c does not have a matching arc from state 0 in the first input graph, so it is ignored.



(h) The next state pair in the queue is $(1, 1)$. There are no arcs leaving state 1 in the first input graph, and $(1, 1)$ is not an accept state in the intersected graph, so it is a dead end. We can remove $(1, 1)$ and its incoming arcs from the intersected graph.



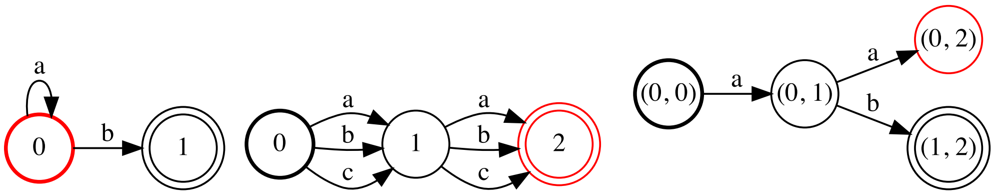
(i) The next state pair in the queue is $(0, 2)$. Again, there are no matching arcs leaving this state pair, and $(0, 2)$ is not an accept state in the intersected graph, hence it is a dead end. We can also remove $(0, 2)$ and its incoming arcs from the intersected graph.

Algorithm 4.1 Intersect

Input: Acceptors $\mathcal{A}_1$ and $\mathcal{A}_2$
Initialize the queue Q and the intersected graph $\mathcal{I}$.
for s_1 and s_2 in all start state pairs of $\mathcal{A}_1$ and $\mathcal{A}_2$ **do**
    Add (s_1, s_2) to Q and as a start state in $\mathcal{I}$.
    if s_1 and s_2 are accept states **then**
        Make (s_1, s_2) an accept state in $\mathcal{I}$.
    end if
end for
while Q is not empty **do**
    Remove the next state pair (u_1, u_2) from Q.
    for all arcs pairs a_1 and a_2 leaving u_1 and u_2 with matching labels **do**
        Get destination states v_1 of a_1 and v_2 of a_2.
        if not yet seen (v_1, v_2) **then**
            Add (v_1, v_2) as a state to $\mathcal{I}$ and to Q.
            if v_1 and v_2 are accept states **then**
                Make (v_1, v_2) an accept state in $\mathcal{I}$.
            end if
        end if
        Get the label ℓ of a_1.
        Get the weights w_1 of a_1 and w_2 of a_2.
        Add an arc from (u_1, u_2) to (v_1, v_2) with label ℓ and weight $w_1 + w_2$.
    end for
end while
Return: The intersected graph $\mathcal{I}$.

does not have any useless states. In the standard terminology, a state is said to be *accessible* if it can be reached from a start state and *coaccessible* if an accept state can be reached from it. A graph is trim if every state is both accessible and coaccessible.

Every state in the intersected graph $\mathcal{I}$ will be accessible, but not necessarily coaccessible, so $\mathcal{I}$ may not be trim. However, the graph will still be correct, so this is primarily an issue of representation size. With some additional work the constructed graph can be kept trim during the operation of the algorithm. Another alternative is to construct a trim graph from the non-trim graph as a post-processing step. Note, in figure 4.2 showing some steps of the intersect algorithm, we removed dead-end states in the intersected graph for clarity; however, algorithm 4.1 does not do this.



(j) The next state pair in the queue is $(1, 2)$. There are no matching arcs for this state pair, but it is an accept state in the intersected graph, so we have to keep it. At this point the queue is empty. The algorithm terminates, and we are left with the complete intersected graph.

Figure 4.2: An illustration of some of the steps (a-j) taken in the intersect algorithm. The two input graphs are on the left and the middle and the construction of the intersected graph is shown on the right.

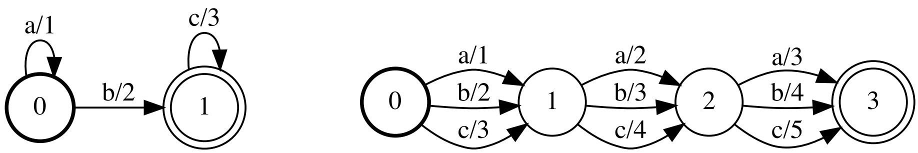


Figure 4.3: The automata on the left recognizes a^*bc^*, the automata on the right recognizes all three letter combinations of the alphabet $\{a, b, c\}$.

Example 4.1. Compute the intersection of the two graphs in figure 4.3. Make sure to update the arc weights correctly in the intersected graph.

The intersection is given in figure 4.4. The intersected graph accepts the sequences *aab*, *abc*, and *bcc* which are the only sequences accepted by both inputs.

■

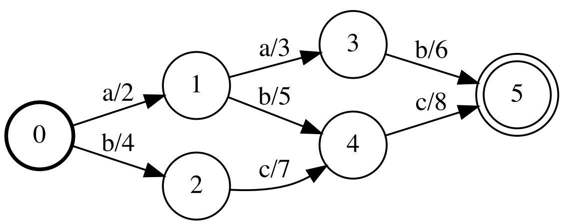


Figure 4.4: The intersection of the graphs in figure 4.3 accepts the strings *aab*, *abc*, and *bcc*.

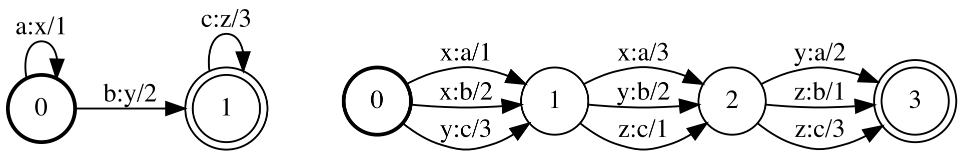


Figure 4.5: Two transducers for which we would like to compute the composition.

4.2 Compose

The composition is a straightforward generalization of the intersection from acceptors to transducers. Assume the first input graph transduces the sequence $\mathbf{x}$ to the string $\mathbf{y}$ and the second graph transduces $\mathbf{y}$ to $\mathbf{z}$. Then the composed graph transduces $\mathbf{x}$ to $\mathbf{z}$.

From an implementation standpoint the compose and intersect algorithms are almost identical. The two minor differences are 1) the way that labels are matched in the input graphs and 2) the labels of the new arcs in the composed graph. Arcs in a transducer have both input and output labels. We match the output arc label from the first graph to the input arc label from the second graph. This means that compose, unlike intersect, is not commutative, since the order of the two graphs makes a difference.

The input label of a new arc in the composed graph is the input label of the corresponding arc in the first input graph. The output label of a new arc in the composed graph is the output label of the corresponding arc in the second input graph. For example, assume we have two arcs, the first with label $x_1 : x_2$, and the second with label $y_1 : y_2$. If $x_2 = y_1$, then the arcs are considered a match. The new arc will have the label $x_1 : y_2$.

Think of matrix multiplication as an analogy or mnemonic device. When multiplying two matrices they have to match on the inner dimension, and the dimensions of the output matrix are the outer dimensions. In the same way the inner labels of the two arcs must match and the resulting arc labels are the outer labels of the two input arcs.

Example 4.2. Compute the composition of the two graphs in figure 4.5.

The composition is given in figure 4.6. As an example, the first input graph transduces *abc* to xyz and the second input transduces xyz to *abb*, *abc*, *bbb*, and *bbc*. Thus, the composed graph should transduce *abc* to all four of *abb*, *abc*, *bbb*, and *bbc*. You can verify this in the graph in figure 4.6.

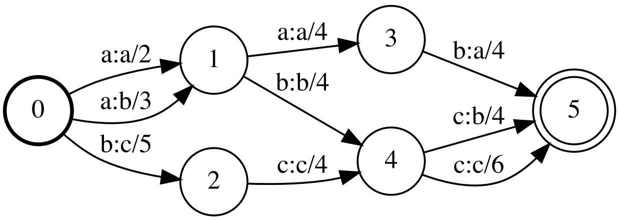


Figure 4.6: The composition of the two graphs from figure 4.5.

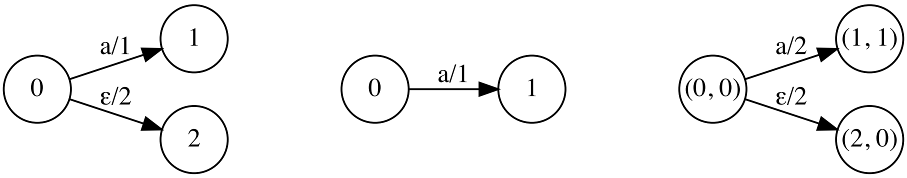


Figure 4.7: The graph on the left has an ϵ transition. Computing the intersection of the left and middle graph results in the graph on the right. The ϵ transition can be followed without consuming an input in the middle graph.

■

4.2.1 Intersect and Compose with ϵ

The basic implementation of intersect and compose we have discussed so far doesn't extend to ϵ transitions. Allowing ϵ transitions in these algorithms makes them more complicated. In this section I will illustrate the challenges with a naive approach and sketch at a high-level how to actually account for ϵ transitions in the intersect and compose algorithms.

First, consider the simpler case when only the first input graph has ϵ transitions. In this case, whenever we encounter an outgoing ϵ transition from a state in the first graph, we can optionally traverse it without matching a corresponding arc in the second graph.

Consider the sub-graphs in figure 4.7. Suppose we are currently looking for outgoing arcs with matching labels from the state $(0, 0)$. When we find a matching pair, we add the new state to the intersected graph and add a corresponding arc. In the ϵ-free case, we only explore the arc's with label a. The state $(1, 1)$ is added to the intersected graph and the queue to be explored. The state $(0, 0)$ is connected to the state $(1, 1)$ with an arc with label a and weight 2.

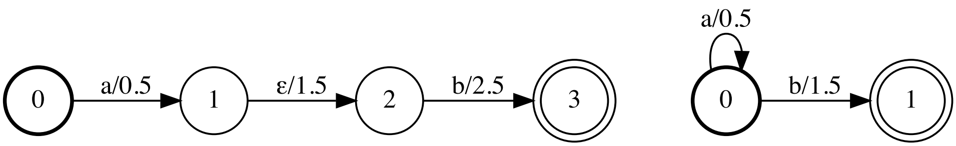


Figure 4.8: An example of two acceptors for which we would like to compute the intersection. The acceptor on the left has an ϵ transition.

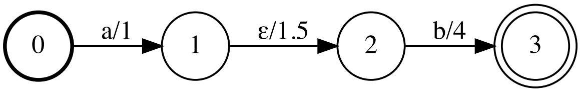


Figure 4.9: The intersection of the two graphs in figure 4.8.

Since state 0 in the first graph has an outgoing ϵ, we can optionally traverse it without traversing any arc in the second graph. In this case, we add the state $(2, 0)$ to the intersected graph and to the queue to be explored. We also add an arc from the state $(0, 0)$ to the state $(2, 0)$ in the intersected graph with a label ϵ and a weight of 2.

Example 4.3. Compute the intersection of the two graphs in figure 4.8.

The intersected graph is in figure 4.9. The ϵ transition is included for clarity, though it could be removed and states 1 and 2 collapsed yielding an equivalent graph.

■

The trickier case to handle is when both graphs have ϵ transitions. If we optionally explore outgoing ϵ arcs in each graph, then we will end up with too many paths in the intersection. Suppose we are given the two graphs in figure 4.10, each of which has an ϵ transition.

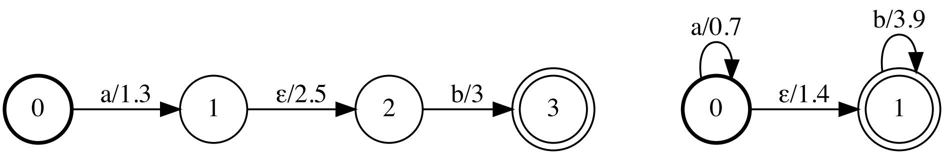


Figure 4.10: Two acceptors, both of which have ϵ transitions.

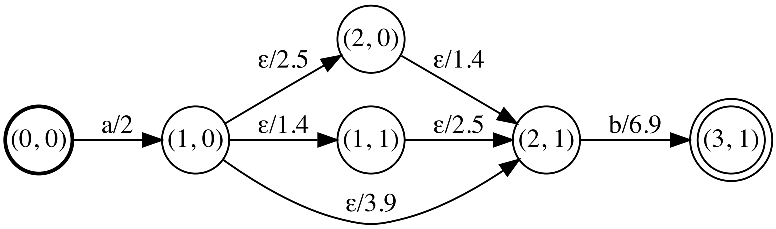


Figure 4.11: An incorrect composition of the two graphs in figure 4.10. Naively following ϵ transitions as we encounter them when computing the intersection results in an incorrect graph. The graph admits too many paths for the sequence ab, and hence the score of that sequence is incorrect.

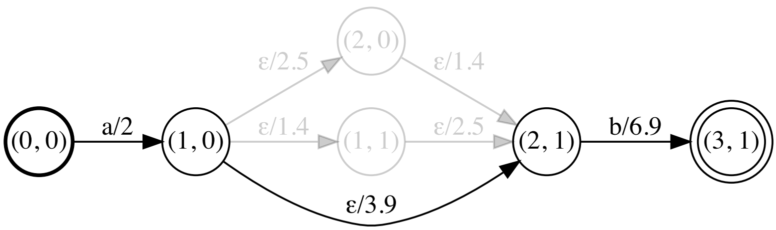


Figure 4.12: Correctly accounting for ϵ transitions when intersecting the two graphs in figure 4.10. Notice that only one of the three resultant paths should be retained in the intersected graph.

If we compute the intersection of the two graphs in figure 4.10, optionally following ϵ transitions as we encounter them, then we end up with the graph in figure 4.11.

The language of this graph is correct. It accepts the string ab which is the only string in the intersection. However, the weight it assigns to the string ab is incorrect. Each individual path has the correct weight, but there are three paths for ab. The final weight will receive three contributions, one from each path, instead of a single contribution from one path. The solution to this problem is to choose only one of the three paths and avoid the inadvertent redundancy. For example we could keep the bottom path and ignore the top two as in the graph in figure 4.12.

4.3 Forward and Viterbi

The forward score and the Viterbi score take a graph as input and return a single scalar result. The *forward score* is the accumulation of the weights of all possible paths from any start state to any accept state in the graph. The weight of the

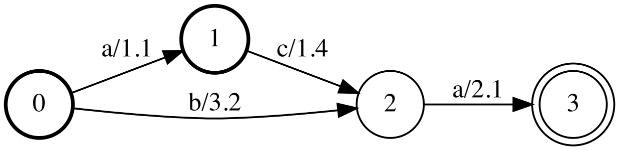


Figure 4.13: An acceptor with three paths from the start states 0 and 1 to the accept state 3.

highest scoring path is the *Viterbi score*, and the path itself is the *Viterbi path*.

> Shortest Distance
>
> In some descriptions of weighted automata, the forward and Viterbi score are introduced as shortest distance algorithms under a respective semiring. The forward score corresponds to the log semiring and the Viterbi score corresponds to the tropical semiring. This is a more general perspective, and useful if you intend to use other semirings. However, we only need the log and tropical semirings in all of the applications we study, so I will restrict the description to the more specific forward and Viterbi score.

Let's start with a couple of examples to show exactly what we are trying to compute, then we will go through a more general algorithm for forward and Viterbi scoring. For the forward and Viterbi score, we will restrict the graphs to be acyclic, meaning no self-loops or cycles. Under certain technical conditions a graph with cycles can admit a computable forward and Viterbi score, but we won't discuss these cases as they don't come up often in machine-learning applications.

The graph in figure 4.13 has three possible paths from the start states to the accept state. The paths and their scores are:

- State sequence $0 \rightarrow 1 \rightarrow 2 \rightarrow 3$ with score 4.6
- State sequence $0 \rightarrow 2 \rightarrow 3$ with score 5.3
- State sequence $1 \rightarrow 2 \rightarrow 3$ with score 3.5

The Viterbi score is the maximum over the individual path scores, in this case $\max\{4.6, 5.3, 3.5\} = 5.3$. The Viterbi path is the sequence of labels which correspond to the arcs contributing to the Viterbi score. We represent the Viterbi path as a simple linear graph as in figure 4.14. Note the Viterbi path may not be unique–multiple paths could all attain the Viterbi score. The forward score is the *log-sum-exp* over all the path scores, in this case $\mathrm{LSE}(4.6, 5.3, 3.5) = 5.81$. The

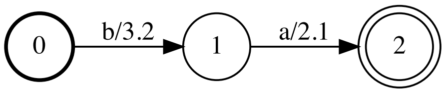


Figure 4.14: The Viterbi path for the graph in figure 4.13. The score of the Viterbi path is the Viterbi score, in this case 5.3.

forward score will always be larger than the Viterbi score. However, the two will converge as the difference between Viterbi score and the second highest scoring path increases.

We computed the forward and Viterbi score by listing all possible paths, computing their individual weights, and then accumulating either with the max or the LSE operations. This approach won't scale to larger graphs because the number of paths can grow combinatorially. Instead, we will use a much more efficient dynamic programming algorithm which works for both the forward and Viterbi score.

The dynamic programming algorithm relies on the following recursions. Consider a state v and let e_i for $i = 1, \ldots, k$ be the set of arcs for which v is the destination node. For a given arc e we let source(e) denote the source node for that arc. The score of all paths which start at a start state and terminate at node v can be constructed from the score of all paths which start at a start state and terminate at the state source(e_i) and the arc weight $w(e_i)$ for $i = 1 \ldots, k$. For the Viterbi score, the recursion is:

$$s_v = \max_{i=1}^{k} \left(s_{\text{source}(e_i)} + w(e_i) \right),$$

where s_v is the score of all paths starting at a start state terminating at state v, and $s_{\text{source}(e_i)}$ is the score of all paths starting at a start state terminating at state source(e_i). The recursion is shown graphically in figure 4.15. In figure 4.15, the Viterbi score of state 3 is the maximum over the weight plus the source node score for all incoming arcs.

The overall Viterbi score is the max of the Viterbi scores over the accept states, $\max_a s_a$ where a is an accept state.

The forward score uses the exact same recursion but with an LSE in place of the max:

$$s_v = \operatorname*{LSE}_{i=1}^{k} \left(s_{\text{source}(e_i)} + w(e_i) \right),$$

and the final score is $\text{LSE}_a \, s_a$.

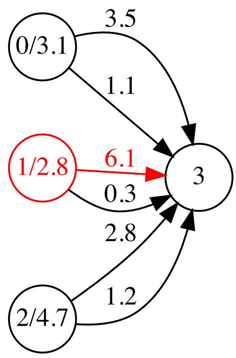


Figure 4.15: A graphical depiction of the recursion for computing the Viterbi score.

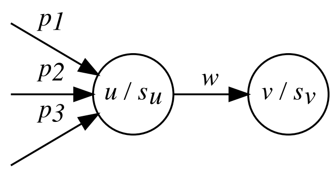


Figure 4.16: The recursion in computing the Viterbi and forward score works because the score s_v at state v can be computed from the weight w and the score s_u at state u.

For both the forward and Viterbi score, the recursion works because the LSE and max operations admit a simple decomposition of the score for all paths terminating at a given state. Suppose, as in the graph in figure 4.16, we have a state v for which we want to compute the score s_v. Suppose also that v has only one incoming arc from state u. Three paths from a start state terminate at state u with the given scores p_1, p_2, and p_3. We can extend each of the three paths from u to v by adding the arc between them, so there are three paths terminating at v as well.

First, suppose we want to compute the Viterbi score at v. The Viterbi score is the maximum of the weights of all three paths terminating at v, namely $s_v = \max\{w + p_1, w + p_2, w + p_3\}$. We can also compute the Viterbi score s_v from the Viterbi score, s_u, of paths terminating at u. In this case $s_v = w + s_u$. This is the recursion we used above but for simplicity with only one incoming arc to v.

The two ways of computing s_v are equivalent:

$$\begin{aligned} s_v &= w + s_u \\ &= w + \max\{p_1, p_2, p_3\} \\ &= \max\{w + p_1, w + p_2, w + p_3\} \\ &= s_v. \end{aligned}$$

The same decomposition works for the forward score and the LSE operation:

$$\begin{aligned} s_v &= w + s_u \\ &= w + \log\left(e^{p_1} + e^{p_2} + e^{p_3}\right) \\ &= \log e^w + \log\left(e^{p_1} + e^{p_2} + e^{p_3}\right) \\ &= \log e^w \left(e^{p_1} + e^{p_2} + e^{p_3}\right) \\ &= \log\left(e^{w+p_1} + e^{w+p_2} + e^{w+p_3}\right) \\ &= s_v \end{aligned}$$

For simplicity, we assume only three paths terminating at u and one arc incoming to v. The argument is easily extended to an arbitrary number of paths and arcs.

4.3.1 Viterbi Path

A Viterbi path is a path for which the Viterbi score is attained. We can compute one of the Viterbi paths with a straightforward extension of the Viterbi scoring algorithm. At each state when we compute the score, we also maintain a back-pointer to the arc which resulted in the maximum score. When the algorithm terminates, we can trace the back-pointers to the start state and extract the Viterbi path.

An example of this can be seen in the graph in figure 4.17.

Each state in the graph is labeled with the Viterbi score over all paths terminating at that state. The red arcs are the arcs which result in the maximum score for the state that they point to. In order to compute the Viterbi path, we trace the red arcs back from the accept state. In this case the Viterbi path state sequence is $0 \to 2 \to 4 \to 6$ with arc labels b, c, and b.

Example 4.4. Compute the Viterbi path of the graph in figure 4.18.

The Viterbi path is shown in figure 4.19. ■

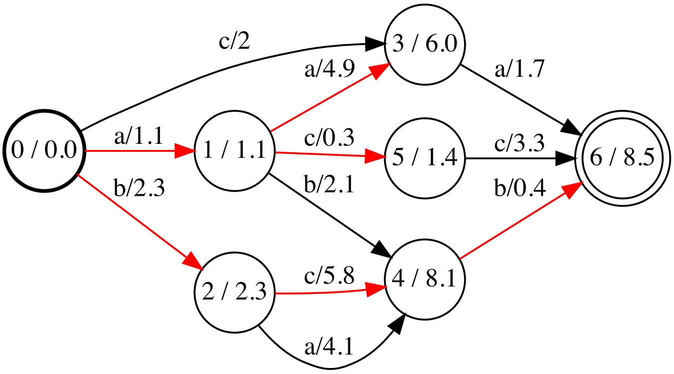


Figure 4.17: Each state is labeled with the state label and Viterbi score from the start state up to that state. The red arrows indicate the arc which is part of the Viterbi path up to the given state. The complete Viterbi path can be found by following red arcs back from the accept state.

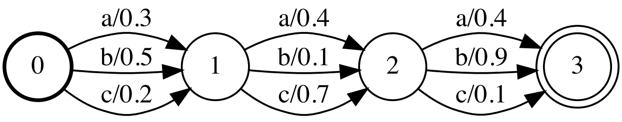


Figure 4.18: An example acceptor for which we would like to compute the Viterbi path.

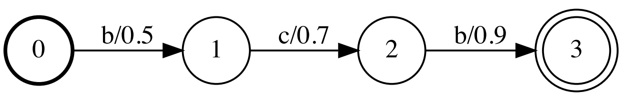


Figure 4.19: The Viterbi path of the graph in figure 4.18.